S-matrix representation of the finite temperature propagator in $\lambda\varphi^4$-QFT

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Abstract

The two-point Green function of the massive scalar $(3+1)$-quantum field theory with $\lambda\varphi^4$ interaction at finite temperature is evaluated up to the 2nd order of perturbation theory. The everaging on the vacuum fluctuations is separated from the everaging on the thermal fluctuations explicitly. As a result, the temperature dependent part of the propagator is expressed through the scattering amplitudes. The obtained expression is generalized for higher orders of perturbation theory.

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1 Introduction

In 1969 R. Dashen, S. Ma and H.J. Bernstein have suggested [1] a generalization of the Beth–Uhlenbeck formula. They have obtained the complete virial expansion for the grand potential Ω where the n-th virial coefficient was expressed in terms of special traces of the $n \to n$ S-matrix elements

$$\beta(\Omega - \Omega_0) = \frac{1}{4\pi i} \sum_{n=2}^{\infty} z^n \int_{n \cdot m}^{\infty} dE \, e^{-\beta E} \left(\mathrm{Tr}_n \, AS^{-1} \frac{\overleftrightarrow{\partial}}{\partial E} S \right)_c, \tag{1}$$

($T = 1/\beta$ is the temperature, z is the activity, S is the scattering matrix, A is the exchange operator, m is the mass of a particle and Ω_0 is the grand potential of the free particle system). The compact and invariant form of the eq.(1) was the reason to claim [1] the validity of this representation also in the relativistic case. We share this point of view in spite of absence of any direct quantum field theory derivations.

In 70-th we exploited this formalism – so called S-matrix formulation of statistical mechanics – for the phenomenological investigation of the hot hadron matter thermodynamics [2–5]. The equation of state derived on the basis of Regge phenomenology for the scattering amplitudes appeared to be essentially nonideal at GeVs temperatures. We have used it for some astrophysical and cosmological applications [6–8]. In particular due to a Van-der-Waals-like nonideality of our equation of state we have obtained in the framework of the Fridman model the exponential expansion of the Universe in the vicinity of the phase transition, providing for a solution the of horizon, flatness, isotropy, primary fluctuations and some others problems. A few years later Guth published [9] his outstanding Inflationary Universe scenario based on a quantum field theory approach.

At present the general fashion for the theoretical treatment of hadron matter (quark-gluon plasma) is ingenuous QCD: perturbation theory, lattice simulations. Nevertheless we believe that the S-matrix approach to the problem is fruitful at least in a qualitative and heuristic sense (as for quantitative analysis see for example ref.[10]). To be more acceptable S-matrix approach needs of course in detailed QFT tests. The perturbation theory analysis is a step in this direction.

In this work we carry out two-loop calculations for the temperature dependent part of the 1PI two-point Green function of $\lambda \varphi^4$ QFT – the simplest but nontrivial quantum field model. We find that it can be represented through the thermal everages of corresponding renormalized scattering amplitudes.

2 The model

We considered a scalar field $\varphi(x)$ in a box with the volume V at the temperature $T = \beta^{-1}$. The points of the Euclidian space-(imaginary)time are enumerated by coordinates $x = (x_0, \vec{x})$. The scalar product is

$$(xy) = x_0 y_0 + (\vec{x}\vec{y}), \qquad x^2 = x_0^2 + \vec{x}^2.$$

The field $\varphi(x)$ is periodical along the time direction $\varphi(x_0, \vec{x}) = \varphi(x_0 + \beta, \vec{x})$ according to Bose-Einstein statistics. Space boundary conditions are irrelevant in thermodynamical limit $V \to \infty$.

The action is

$$S[\varphi] = \int d^3x \int_0^\beta dx_0 (\mathcal{L}_0(\varphi) + \mathcal{L}_I(\varphi))$$

where the free-field Lagrangian is

$$\mathcal{L}_0(\varphi) = \frac{1}{2}\varphi(x)(m_0^2 - \Box)\varphi(x),$$

$$\Box = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

The interaction due to

$$\mathcal{L}_I(\varphi) = \frac{\lambda_0}{4!}\varphi^4(x),$$

where m_0, λ_0 are the bare mass and the coupling constant. The generating functional is

$$\exp\{Z[J]\} = \int D\varphi e^{-S[\varphi] - (J,\varphi)},$$

where $J(x)$ is the external source and

$$(J, \varphi) = \int d^3x \int_0^\beta dx_0 J(x)\varphi(x).$$

The Green functions are defined as usual [11,12]

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)}.$$

The one particle irreducible (1PI) Green functions are

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n W[\psi]}{\delta \psi(x_1) \ldots \delta \psi(x_n)},$$

where

$$\psi(x) = \frac{\delta Z[J]}{\delta J(x)}, \qquad \frac{\delta W[\psi]}{\delta \psi(x)} = -J(x).$$

We use the Fourier transformation (direct and inverse) in the appropriate normalization

$$f(x_0, \vec{x}) = \frac{T}{(2\pi)^3} \sum_{p_0} \int d^3p\, e^{-ip_0 x_0 - i\vec{p}\vec{x}} \widetilde{f}(p_0, \vec{p}),$$

$$\widetilde{f}(p_0, \vec{p}) = \int d^3x \int_0^\beta dx_0 e^{ip_0 x_0 + i\vec{p}\vec{x}} f(x_0, \vec{x}),$$

$$p_0 = 2\pi \ell T, \qquad \ell = 0, \pm 1, \pm 2, \ldots.$$

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3   1PI $\Gamma^{(2)}$-function

At $T = 0$ and $V \to \infty$ the Green functions depend on momenta only. At $T \neq 0$ they depend also on T. The $\Gamma^{(2)}$-function is the inverse propagator: $G^{(2)} \cdot \Gamma^{(2)} = 1$. The observable corresponding to the propagator is the space (equal-time) two-point correlation function

$$\kappa(|\vec{x}|) = \frac{T}{(2\pi)^3} \sum_{p_0} \int \frac{d^3 p\, e^{-i\vec{p}\vec{x}}}{\Gamma^{(2)}(p_0^2, \vec{p}^2, T)}. \tag{2}$$

The correlation length $x_c(T)$ is determined by the nearest to the real axes zero of the denominator of integrand (2) in the complex plane $|\vec{p}|$ at $p_0 = 0$.

$$\Gamma^{(2)}(0, -x_c^{-2}(T), T) = 0.$$

At $T = 0$ the correlation length is simply the inverse mass of the boson and at $T \neq 0$ it defines the Debye screening.

   The goal of this paper is to separate the vacuum (quantum) fluctuations from thermal ones. Let us remind the well known one-loop result for $\Gamma^{(2)}$-function

$$\Gamma^{(2)}(p_0^2, \vec{p}^2, T) = p^2 + m_0^2 + \frac{\lambda}{2} \frac{1}{(2\pi)^4} \int \frac{d^4 q}{q^2 + m^2} + \frac{\lambda}{(2\pi)^3} \int \frac{d^3 q}{2\omega(e^{\beta\omega} - 1)}, \tag{3}$$

where

$$p^2 = p_0^2 + \vec{p}^2, \qquad \omega^2 = \vec{p}^2 + m^2.$$

This expression clarifies what we mean in "vacuum" and "thermal" fluctuations. We call the fist integral in (3) "vacuum loop" and the second one – "thermal loop". The full loop is divided in two terms – additively in the first order of perturbation theory

Thermal fluctuations exponentially vanish at $T \to 0$ and

$$\Gamma^{(2)}(p_0^2, \vec{p}^2, T \equiv 0) = \Gamma^{(2)}(p^2)$$

where $\Gamma^{(2)}(p^2)$ is the usual inverse propagator. Let us define the temperature dependent part by extracting from the full propagator its value at zero temperature

$$\gamma(p_0^2, \vec{p}^2, T) = \Gamma^{(2)}(p_0^2, \vec{p}^2, T) - \Gamma^{(2)}(p^2). \tag{4}$$

Since the $2 \to 2$ scattering amplitude in the 1st order of PT is

$$A^{(2)}(p, q; p', q') = -\lambda$$

the γ-function can be written as

$$\gamma(p_0, 2, \vec{p}^2, T) = -\frac{1}{(2\pi)^3} \int \frac{d^3 q\, A^{(2)}(p, q; p, q)}{2\omega(e^{\beta\omega} - 1)} \tag{5}$$

Of course, this expression seems artificial for the trivial 1st order of PT but it shows what we search for higher orders.

4   The second order of PT

The calculations at $T \neq 0$ are more complicated in 2nd order compared to that at $T = 0$ by two reasons: the regularization (cut off) is not symmetric in momentum space and there is a loop summation instead of loop integration. The last obstacle is important because at finite temperature there appears loop divergences depending on T. They come from different diagrams. One of them contains a simple sum and the other – the double sum. So care should be taken when transforming sums to integrals to cancel the above mentioned divergences.

The $\Gamma^{(2)}$-function in the 2nd order is saturated by following diagrams

$$\text{—}\bigoplus\text{—} \;=\; \text{——} \;+\; \frac{\lambda_0}{2}\;\text{——}\bigcirc\text{——} \;-\; \frac{\lambda_0^2}{4}\;\text{——}\text{——} \;-\; \frac{\lambda_0^2}{6}\;\text{——}\bigcirc\text{——}$$

(6)

and 1PI $\Gamma^{(4)}$-function

$$\bigotimes \;=\; -\lambda_0\;\bigtimes \;+\; \frac{\lambda_0^2}{2}\left(\; \bowtie \;+\; \text{—}\text{—} \;+\; \text{—}\bowtie \;\right).$$

(7)

The appropriate regularization is $|\vec{p}| \leq \Lambda$, i.e. it is a cylinder in momentum space rather then a sphere as usual at $T = 0$. Loop summations over zero's components of momenta are convergent by itself. Subtraction points are defined at $T = 0$ as follows

$$\Gamma^{(2)}(p^2)\Big|_{p^2=-m} = 0, \qquad \frac{\partial \Gamma^{(2)}(p^2)}{\partial p^2}\Big|_{p^2=0} = 1, \tag{8}$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4)\Big|_{p_i=0} = -\lambda, \tag{9}$$

where m and λ are renormalized (observable), mass and coupling constant.

The technical problem is to transform the loop sums into integrals. For the simple sum over q_0 the answer is [12]

$$S\nu = T\sum_{q_0} \frac{1}{(q^2+m^2)^\nu} = \frac{1}{2\pi}\int \frac{dq_0}{(q^2+m^2)^\nu}\left(1 + \frac{2}{e^{-i\beta q_0}-1}\right), \nu = 1, 2, \tag{10}$$

or

$$S_1 = \frac{1}{2\pi}\int \frac{dq_0}{Q^2+m^2} + \frac{1}{\omega(e^{\beta\omega}-1)}, \tag{11}$$

$$S_2 = \frac{1}{2\pi}\int \frac{dq_0}{Q^2+m^2} + \frac{1}{\omega}\frac{\partial}{\partial\omega}\left(\frac{1}{2\omega(e^{\beta\omega}-1)}\right). \tag{12}$$

There is more trouble with double sum S_3 contained in the sunrise diagram $\text{—}\bigcirc\text{—}$,

$$S_3 = T^2 \sum_{q_0 k_0} \frac{1}{(q^2 + m^2)(k^2 + m^2)[(q + k - p)^2 + m^2]}. \tag{13}$$

The substitution (10) is not correct in this case owing to the pole surface in the complex manifold $q_0 \otimes k_0$ caused by the factor $[(q + k - p)^2 + m^2]^{-1}$ in (13). Careful manipulations give the result

$$S_3 = \frac{1}{(2\pi)^2} \int \frac{dq_0 dk_0}{(q^2 + m^2)(k^2 + m^2)[(q + k - p)^2 + m^2]} +$$

$$+ \frac{3}{2\omega_k(e^{\beta\omega_k} - 1)} \frac{1}{2\pi} \int \frac{dq_0}{(q^2 + m^2)} \left(\frac{1}{(q + k - p)^2 + m^2} + \frac{1}{(q - k - p)^2 + m^2} \right) +$$

$$+ \frac{3}{2\omega_k(e^{\beta\omega_k} - 1)2\omega_q(e^{\beta\omega_q} - 1)} \times \left(\frac{1}{(k + q + p)^2 + m^2} + \right.$$

$$\left. + \frac{1}{(k - q + p)^2 + m^2} + \frac{1}{(q - k + p)^2 + m^2} + \frac{1}{(k + q - p)^2 + m^2} \right) \tag{14}$$

The second diagram in the r.h.s. of (6) is renormalized in the 2nd order of PT with the help of eqs. (6)–(9) i.e.

$$\lambda_0 = \lambda + \frac{3\lambda^2}{2} \frac{1}{(2\pi)^4} \int \frac{d^4q}{(q^2 + m^2)^2},$$

$$m_0^2 = m^2 - \frac{\lambda}{2} \frac{1}{(2\pi)^4} \int \frac{d^4q}{q^2 + m^2}.$$

Therefore up to the power λ^2 we have

$$\frac{\lambda_0}{2} \;\;\bigcirc\!\!-\!\!\bullet\!\!-\;\; = \frac{\lambda_0}{2} T \sum_{q_0} \frac{1}{(2\pi)^3} \int \frac{d^3q}{q^2 + m_0^2} = \frac{\lambda}{2} T \sum_{q_\nu} \frac{1}{(2\pi)^3} \int \frac{d^3q}{q^2 + m^2} +$$

$$+ \frac{\lambda^2}{4} \left(T \sum_{q_0} \frac{1}{(2\pi)^3} \int \frac{d^3q}{(q^2 + m^2)^2} \right) \cdot \left(\frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} \right) + \tag{15}$$

$$+ \frac{3\lambda^2}{4} \left(\frac{1}{(2\pi)^4} \int \frac{d^4q}{(q^2 + m^2)^2} \right) \cdot \left(T \sum_{k_0} \frac{1}{(2\pi)^3} \int \frac{d^3k}{k^2 + m^2} \right).$$

$$\frac{\lambda_0^2}{4} \;\;\overset{\bigcirc}{\underset{\bullet}{\bigcirc}}\!\!\!-\;\; = \frac{\lambda^2}{4} \left(T \sum_{q_0} \frac{1}{(2\pi)^3} \int \frac{d^3q}{(q^2 + m^2)^2} \right) \left(T \sum_{k_0} \frac{1}{(2\pi)^3} \int \frac{d^3k}{k^2 + m^2} \right). \tag{16}$$

$$\frac{\lambda_0^2}{6} \quad \text{—} \bigcirc \text{—} \quad = \frac{\lambda^2}{6} T^2 \sum_{q_0 k_0} \frac{1}{(2\pi)^6} \int \frac{d^3q\, d^3k}{(q^2+m^2)(k^2+m^2)[(q+k-p)^2+m^2]} \quad (17)$$

Let us write the temperature dependent part of $\Gamma^{(2)}$-function (4) as a sum of two terms

$$\gamma(p_0^2, \vec{p}^2, T) = \gamma_2(p_0^2, \vec{p}^2, T) + \gamma_3(p_0^2, \vec{p}^2, T).$$

Then combining (15)–(17) and regarding transformations from sums to integrals (10)–(14) we obtain

$$\gamma_2(p_0^2, \vec{p}^2, T) = -\frac{1}{(2\pi)^3} \int \frac{d^3q}{2\omega_q(e^{\beta\omega_q}-1)} \left\{ -\lambda + \frac{\lambda^2}{2} \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2+m^2} \times \right.$$

$$\left. \times \left[\frac{1}{(k-q-p)^2+m^2} + \frac{1}{(k+q-p)^2+m^2} - \frac{2}{k^2+m^2} \right] \right\}, \quad (18)$$

where $q_0 = i\omega_q$;

$$\gamma_3(p_0^2, \vec{p}^2, T) = -\frac{1}{2!(2\pi)^6} \int \frac{d^3q\, d^3k}{2\omega_q(e^{\beta\omega_q}-1)2\omega_k(e^{\beta\omega_k}-1)} \lambda^2 \left[\frac{1}{(k+q+p)^2+m^2} + \right.$$

$$\left. + \frac{1}{(k+q-p)^2+m^2} + \frac{1}{(k+p-q)^2+m^2} + \frac{1}{(q+p-k)^2+m^2} + \frac{1}{\vec{q}^2} \right], \quad (19)$$

where $k_0 = i\omega_k$, $q_0 = i\omega_q$.

One can recognize in the braces of (18) the renormalized $2 \to 2$ scattering amplitude at zero angle in the 2nd order of PT. So expected expression (5) is confirmed in 2nd order of PT.

Now we assume by analogy that γ_3 should have the form

$$\gamma_3(p_0^2, \vec{p}^2, T) = -\frac{1}{2!(2\pi)^6} \int \frac{d^3q\, d^3k\, A^{(3)}(p,q,k;\, p,q,k)}{2\omega_q(e^{\beta\omega_q}-1)2\omega_k(e^{\beta\omega_k}-1)} \quad (20)$$

Really, the lowest order in which the $3 \to 3$ scattering amplitude appears is λ^2. There are 10 tree diagrams (the direct channel diagram plus 9 exchanged diagrams) which contribute to the $3 \to 3$ amplitude. Collecting them in 3 subsets as shown below

$$(21)$$

$$(22)$$

$$(23)$$

one can see that the first four terms in square brackets of (19) are just the four diagrams (21) with the allowance $p = p'$, $q = q'$, $k = k'$ (zero angles scattering). More problematic are the diagrams (22). On mass shell of momenta q or k these diagrams diverge themselves and should be regularized. Several papers were devoted to this problem [13]. Our direct field-theory calculations give the simple recipe for such regularization. Namely one must set $p - p' = \epsilon = (\epsilon_o, \vec{\epsilon})$ and take the limit in the final result in the following order

$$\gamma_3(p_0^2, \vec{p}^2, T) = \lim_{\vec{\epsilon}=0}\left(\lim_{\epsilon_0=0}\frac{-1}{2!(2\pi)^3}\int\frac{d^3q\,d^3k\,A^{(3)}(p,q,k;\,p+\epsilon,q,k)}{2\omega_q(e^{\beta\omega_q}-1)2\omega_k(e^{\beta\omega_k}-1)}\right).$$

As to the last two diagrams (23), they do not contribute to γ_3 because the considered $\Gamma^{(2)}$-function by definition should contain 1PI diagrams only.

5 Conclusion

Surprisingly the temperature dependent part of $\Gamma^{(2)}(p_0^2, \vec{p}^2, T)$ – eqs. (5) and (20) – can be expressed in an extremely compact and physically transparent form through the scattering amplitudes. These low order PT calculations give us the opportunity to generalize naturally enough the obtained result in the following way

$$\gamma(p_0^2, \vec{p}^2, T) = \sum_{n=2}^{\infty}\gamma_n(p_0^2, \vec{p}^2, T),$$

$$\gamma_{n+1}(p_0^2, \vec{p}^2, T) = \qquad\qquad\qquad\qquad (24)$$

$$= -\frac{1}{n!(2\pi)^{3n}}\int\left(\prod_{l=1}^{n}\frac{d^3q_l}{2\omega_l(e^{\beta\omega_l}-1)}\right)A^{(n+1)}(p,q_1,\ldots,q_n;\,p,q_1,\ldots,q_n).$$

This expression is linear in the scattering amplitudes and looks like a contribution of so called "first-type" diagrams in the S-matrix formulation of statistical mechanics. We can

not exclude without further analysis the appearence of some bilinear terms in amplitudes in the exact formula for $\gamma(p_0^2, \vec{p}^2, T)$ similar to the so-called "second-type" diagrams. But we shall not be surprised if the eq. (24) satisfies the exact representation.

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